UNIT 4

Trigonometry

The branch of mathematics that deals with the relations of the sides and angles of triangles and with the relevant functions of any angle is known as *trigonometry*.

Background - Simple Triangle Trigonometry

In a geometry class, you probably were first acquainted with the trigonometric functions that represent ratios of legs of right triangles. We will now revisit this topic with the only difference being that we will place these right triangles inside a circle and view them as *reference triangles*. For the point (x, y) on a circle of radius r at an angle of θ, we can define six important functions as the ratios of the sides of the corresponding triangle:

1. $\sin\theta = \dfrac{y}{r}$
2. $\csc\theta = \dfrac{r}{y}$
3. $\cos\theta = \dfrac{x}{r}$
4. $\sec\theta = \dfrac{r}{x}$
5. $\tan\theta = \dfrac{y}{x}$
6. $\cot\theta = \dfrac{x}{y}$

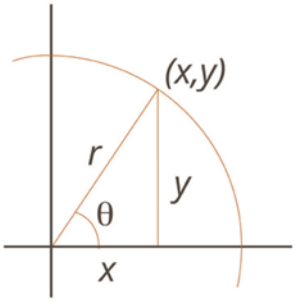


Figure 4.0-1

Note that form (1) and (3) above, we can make the observations that

$$x = r\cos\theta, \text{ and}$$

$$y = r\sin\theta$$

Further, if the reference triangle under consideration is embedded in a *Unit Circle*, a circle of radius 1, then

$$x = \cos\theta, \text{ and}$$

$$y = \sin\theta$$

Also notice that for a triangle centered in a Unit Circle such as in Figure 4.0-1, by the Pythagorean Theorem, we have that

$$x^2 + y^2 = 1$$

Thus, by substitution, one obtains the *Pythagorean Identity*

$$\cos^2\theta + \sin^2\theta = 1$$

[NOTE: $\sin^2\theta$ means $(\sin\theta)^2$ which does not equal $\sin\theta^2$]

Lesson 4.1: Working With Identities

Definition: An *identity* is an equation that is always true. "Always" here means that the equation will be true no matter what numbers you substitute for the variables in the equations.

Exploration 4.1.1: Identifying Identities

1. Is $a(b+c) = ab + ac$ an identity? Why or why not?
2. Is $2a = 6$ an identity? How about $a^2 = -1$? What about $\frac{a}{a} = 1$?

   What about $(a-3)(a-4)(a+2) = a^3 - 5a^2 - 2a + 24$?
3. Is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ an identity? Why or why not?
4. What identities do you know from algebra?
5. Why isn't $a^2 + b^2 = c^2$ an identity? Can you think of *conditions* on *a, b, and c* that make the equation an identity?
6. What kinds of strategies can you use to prove that an equation is an identity?
7. Sometimes we test out equations by substituting in numbers; if we try 10 substitutions for our variables, can we say the equation is an identity? If so, why? If not, then what is the value in testing the equation?
8. What would you have to do to prove that an equation is *not* an identity?

Exploration 4.1.2: More Trigonometric Identities

You have seen that $\cos^2\theta + \sin^2\theta = 1$ is an identity based on the Pythagorean Theorem for a reference triangle inside a Unit Circle.

1. Prove the identity $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$.

2. Create at least three more trigonometric identities based on the couple that you have already learned.

Lesson 4.2: Trigonometric Foundations

A *radian* is a unit of measure for angles that can be used instead of degrees. Just as one complete circle is 360 degrees in angle measure, one complete circle is an angle of 2π *radians* (in other words – a little more than 6 radians will fit on the circumference of a circle when laid side by side). Notice, this implies that a central angle measure of 1 radian in a circle can be subtended by taking the radius of the circle and overlaying it on top of the circumference of the circle. One advantage to using radians is that radians are "dimensionless." That is, radians are simply real numbers that can represent any quantity that we would like to attribute them to in context. Therefore, a half-circle is π radians, and a quarter circle is $\frac{\pi}{2}$ radians.

To convert from radians to degrees or degrees to radians consider the fact that 1 degree = $\frac{\pi}{180}$ radians. Thus, to convert from degrees to radians, multiply the degree measure by $\frac{\pi \text{ radians}}{180^\circ}$. Likewise, to convert from radians to degrees, multiply the radian measure by $\frac{180^\circ}{\pi \text{ radians}}$. Figure 4.2-1 illustrates the *common trigonometric angles* around the Unit Circle as measured in radians.

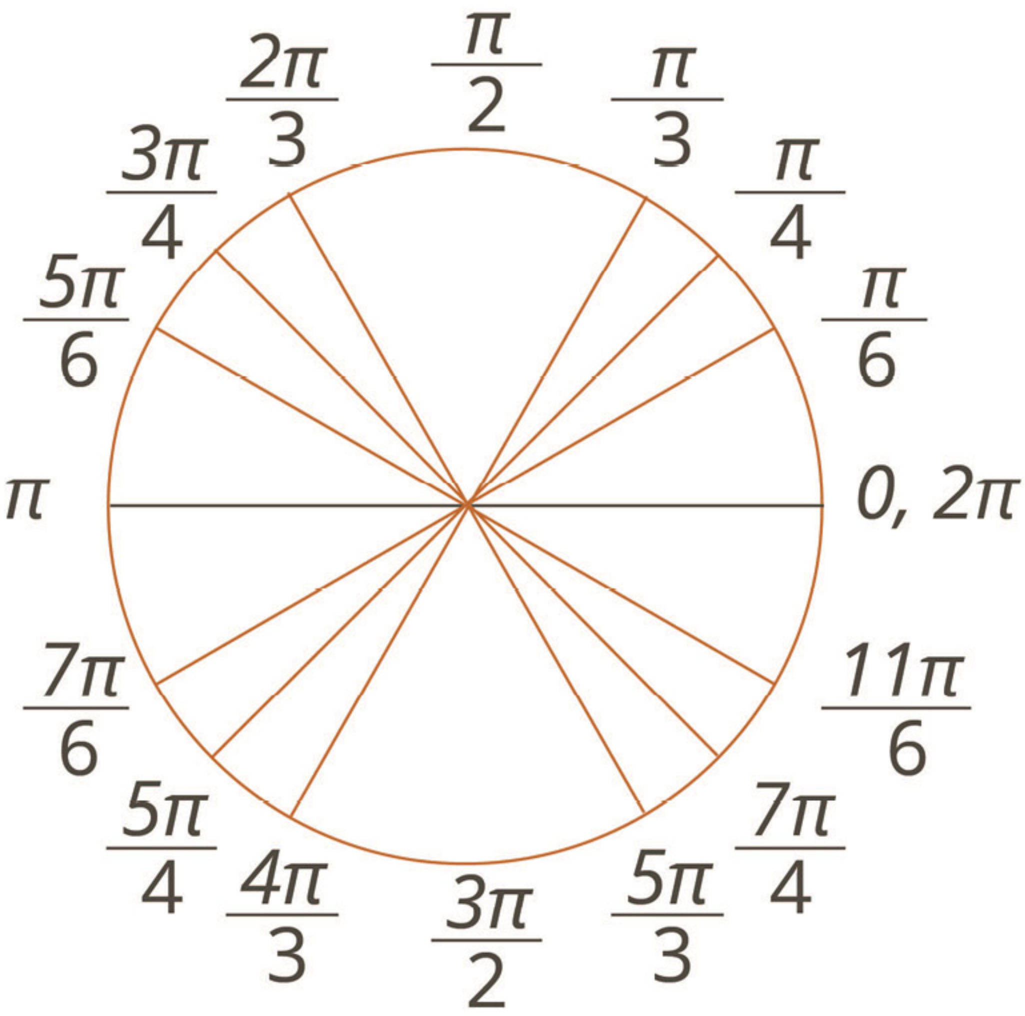


Figure 4.2-1: The Unit Circle

!! Note that there is a real number associated with any angle drawn around the Unit Circle, and not all of these numbers are expressed in terms of π. Most of these *common trigonometric angles* are derived from the fact that they are angles associated with the $30° - 60° - 90°$ or the $45° - 45° - 90°$ right triangle relationships that you learned about in geometry as part of your study of the Pythagorean Theorem. Your instructor may want to take time to reacquaint you with these triangles. They look like this:

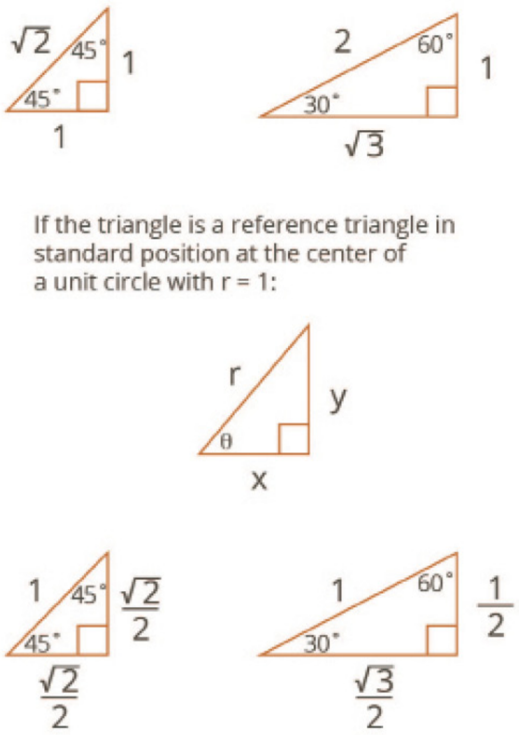
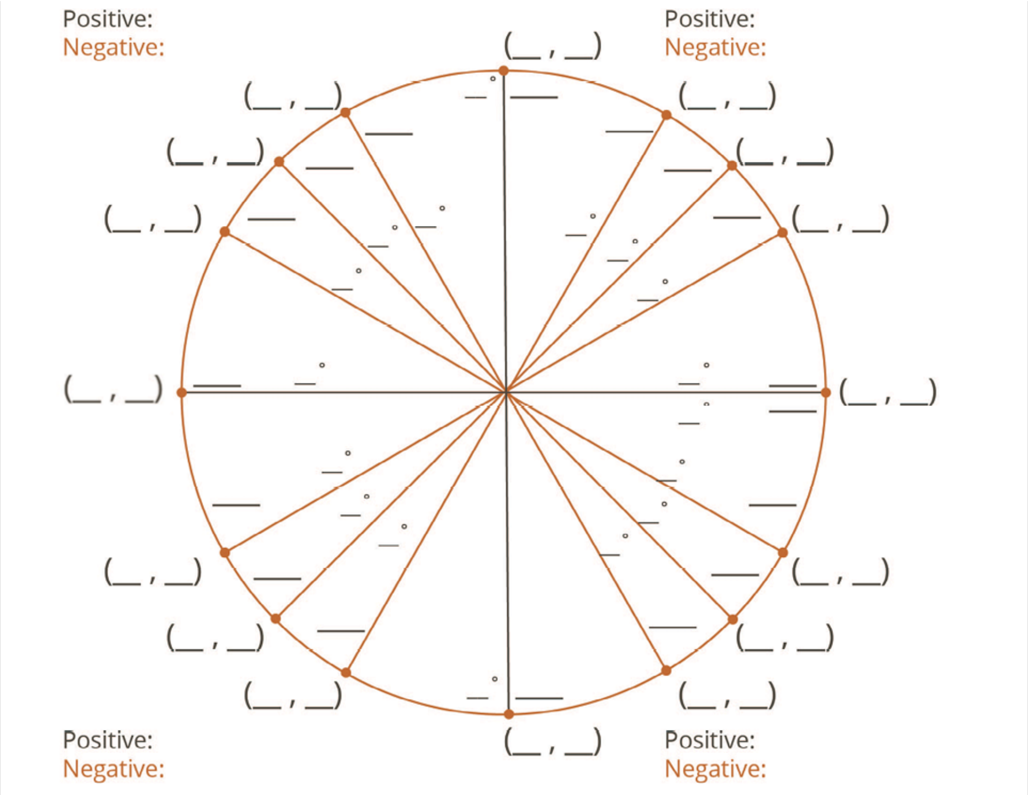

If the triangle is a reference triangle in standard position at the center of a unit circle with r = 1:

Figure 4.2-2: Special Triangles

Definition: For any angle there is an *initial side* and *terminal side*. To view any angle, θ in *standard position*, draw a ray from (0,0) towards (1,0) on the plane, and sweep out θ radians by rotating the ray from around the origin. If θ is positive, sweep in a counterclockwise direction. If θ is negative, sweep in a clockwise direction.

Exploration 4.2.1: Common Angles on the Unit Circle

1. Use the information shown in Figures 4.2-1 and 4.2-2 to fill in the degree measure, radian measure, and coordinate pairs for each of the angles shown in the template below. The goal of this Exploration is to actually work with the right triangle relationships and the degree and radian conversions to convince yourself of how all the information in Figure 4.2-2 was derived. It is important that you take the time to memorize these values as you need to have this information at your fingertips at this point in your mathematical careers.

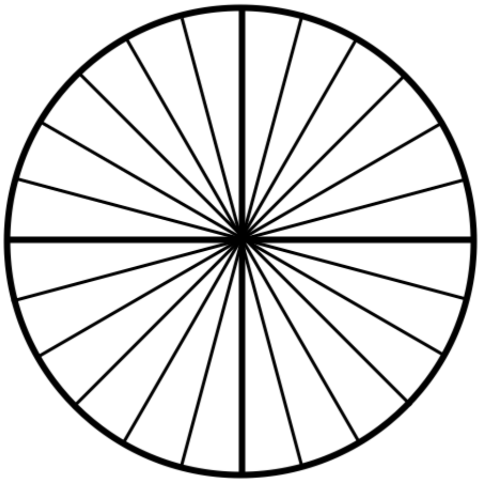


2. Draw in an angle that is not given in the picture. Given that you are working inside a Unit Circle, how could you find the (cosine, sine) ordered pair associated with the angle?

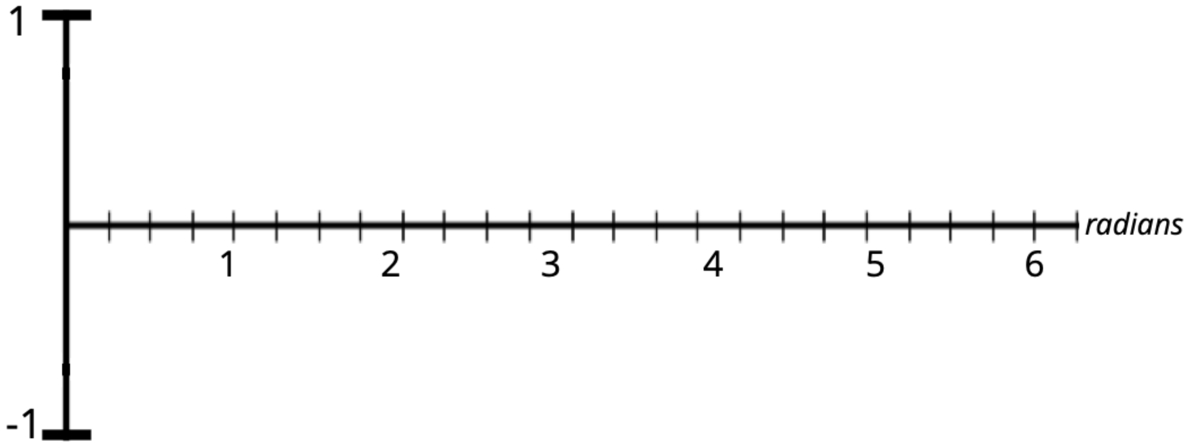
3. What do the angles π and $-\pi$ have in common? Can you think of three more angles that have this same relationship in common?

Lesson 4.3: The Trigonometric Functions Off the Unit Circle

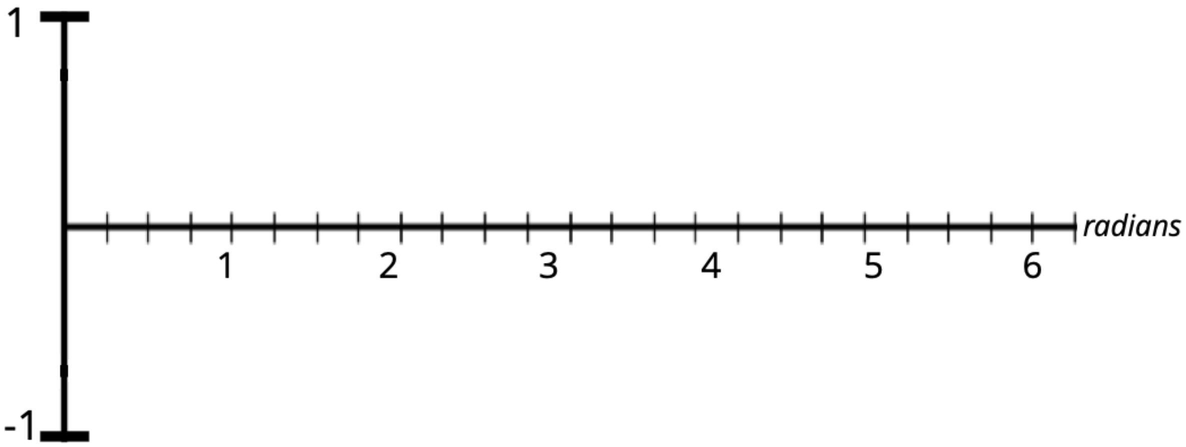
Exploration 4.3.1: Unraveling the Unit Circle



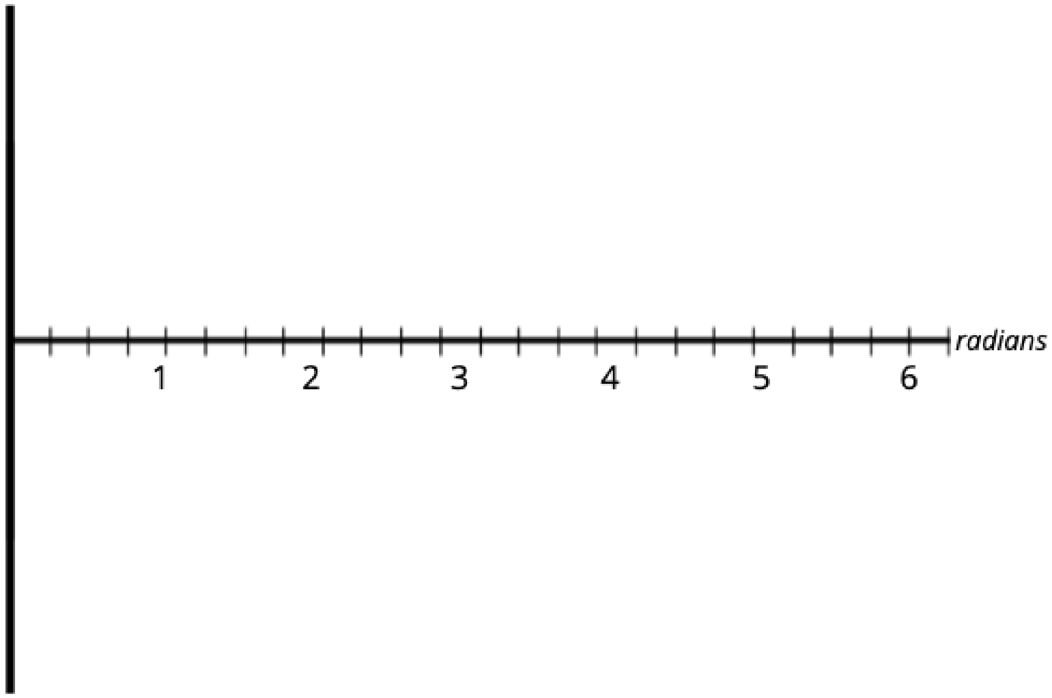
Imagine the above circle with a radius of 1” is a bicycle wheel with 24 spokes. Measure the vertical distance perpendicular from the x-axis of the circle to the end point of each spoke. Record these vertical distances on the provided coordinate plane as y-values. (We will use a negative when the endpoint is below the center). The horizontal spoke will be considered spoke 0. The resulting graph is $y = \sin(x)$. Why is this the case?



Repeat the process, but instead measure the horizontal distance perpendicular from the y-axis of the circle to the end point of each spoke. The resulting graph is $y = \cos(x)$. Why is this the case?

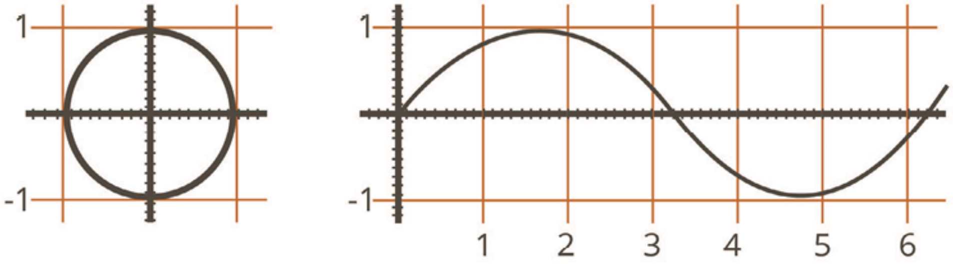


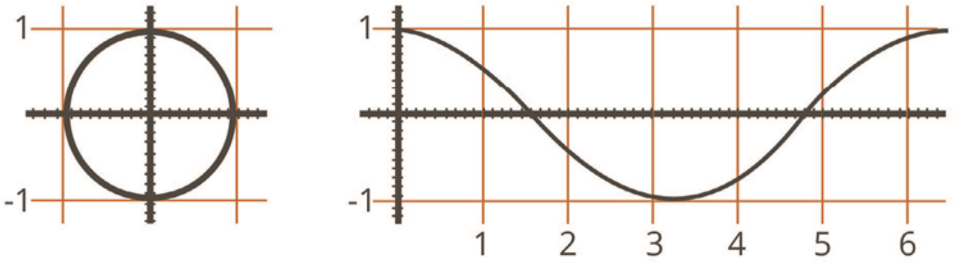
Create a graph of the ratio of the *sine* values to the *cosine* values; that is, the new y value will consist of the y-value of the sine graph divided by the y-value of the cosine graph. Which graph is this?



Exploration 4.3.1 Challenge: More Unraveling - Circles and Other Figures

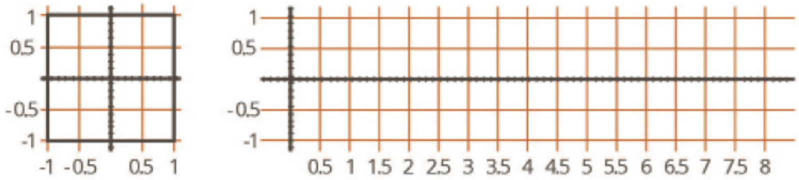
An object moves around a circular track at a constant speed. The object proceeds counter-clockwise starting at the positive intersection of the circumference of the circle and the x-axis. The first graph shows the vertical component (i.e., the distance from the x-axis) as a function of time. The second graph shows the horizontal component (i.e., the distance from the y-axis as a function of time). These results should be obvious to you based on the previous Exploration.


vertical component

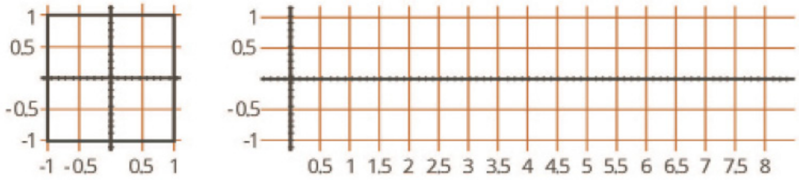

horizontal component

a. An object moves around a square track at a constant speed (1 meter per second). The object proceeds counter-clockwise starting at the positive intersection of an edge of the square and the x-axis. In the graphs below, sketch the vertical and horizontal components of the object as a function of time.

vertical component

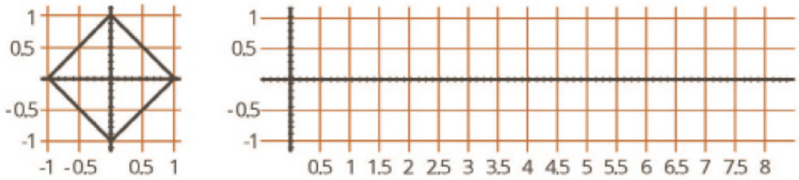


horizontal component

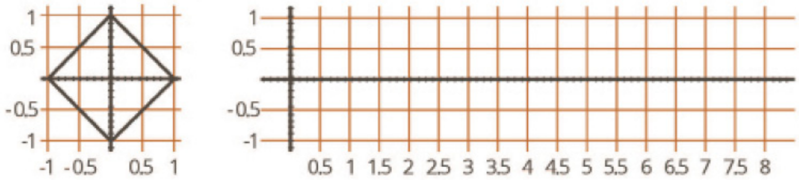


b. An object moves around a diamond-shaped track at a constant speed (1 meter per second). The object proceeds counter-clockwise starting at the positive intersection of a vertex of the diamond and the x-axis. In the graphs below, sketch the vertical and horizontal components of the object as a function of time.

vertical component



horizontal component



Exploration 4.3.2: Find the Domains of Sine, Cosine and Tangent

1. Show that the domain of $f(x) = \sin\ x$ is all real numbers using the definition of the sine function.
2. What is the domain of $f(x) = \cos\ x$? Justify your answer.
3. What is the domain of $f(x) = \tan\ x$? Justify your answer.

Exploration 4.3.3: Dealing with Horizontal Compressions/Stretches and the Period of the Tangent Function

Based on what you have learned, graph the function $y = \tan x$ on graph paper without using a graphing calculator or computer program. Use this as a reference to help you answer the following questions.

1. What are the vertical asymptotes of $y = 2\tan(2x)$?
2. What are the vertical asymptotes of $y = 5\tan(0.1x)$?
3. Generalize: What are the vertical asymptotes of $y = a\tan(bx)$, where a and b are positive constants?
4. Does this answer change if a is negative? What if b is negative?

Sinusoidal Equations: A Summary

One very important thing to notice about the sinusoidal trigonometric functions is that they are *periodic.* The term *period function* refers to the fact that the function values repeat exactly over certain intervals of the domain.

> **Definition:** A function f is called periodic if there exists a positive real number h such that $f(x+h) = f(x)$, for all x in the domain of f. The smallest positive number h is the period of f.

Here are the properties attributed to the constants in the sinusoidal equation:

Given the *sinusoidal* equation

$$y = C + A\cos\left[B(x-D)\right]$$

we first note that we could have also used sine as the base function within the equation. The constants A, B, C, and D can be defined in the following way according to the transformation effect that each has on the function.

1. The C constant in the above equation is called the *sinusoidal axis value* or the vertical shift for the function
$$C = \frac{y_{\text{max}} + y_{\text{min}}}{2}$$
2. The constant A is called the *amplitude* of the function.
$$A = \frac{y_{\text{max}} - y_{\text{min}}}{2}$$
3. The *observed period* is determined by
$$\textit{observed period} = \frac{2\pi}{|B|} \therefore B = \frac{2\pi}{\textit{obs. per.}}$$

Keep in mind that B is NOT the period of the function, but B plays a role in determining the actual period as described above.

4. Last, D in the form of the function given is called the *phase shift* or horizontal shift of the function.

Using this listed information, one could identify the sinusoidal equation that models certain data suspected of being sinusoidal in nature. The sinusoidal trigonometric functions have many applications in that they are used to model many periodic situations that arise in the world around us such as planetary motion, day length, biological systems, sound waves, and energy waves in general.

Exploration 4.3.4: Transformations of Sinusoidal Graphs

1. Plot $y = \cos\, x$ for x in the interval $[-\pi, \pi]$.
2. How might you change the equation of problem 1 to make the entire graph shift $\frac{\pi}{2}$ units to the left?
3. While maintaining the phase shift from problem 2, how might you compress the graph horizontally by a factor of 2? What is the equation for this graph?
4. How might you change the equation of problem 3 to extend the maximum and minimum values of the graph by a factor of 3? What is the equation for this graph?
5. How might you change the equation of problem 4 to make the entire graph shift up 2 units? What is the equation for this graph?

Graph two cycles for each of the supplied functions without a calculator; label both axes with "important" values.

6. $f(x) = 4 + 3\cos\left(\frac{\pi}{6}(x-2)\right)$
7. $f(x) = 1 - 3\cos\left(\frac{\pi}{3}(x-0.5)\right)$
8. $f(x) = -4 + 5\sin\left(\frac{2}{3}\left(x+\frac{\pi}{2}\right)\right)$
9. $f(x) = 1 + 2\sin\left(\frac{2\pi}{3}x - \frac{\pi}{6}\right)$
10. $f(x) = 3 - 2\sin\left(2x + \frac{\pi}{6}\right)$

Exploration 4.3.5: Modeling with Sinusoidal Equations

For each problem situation below, you are to complete the following:

a) Draw a picture of the situation labeling all important points.
b) State the domain and range for the problem situation.
c) Write the function that models the situation and provide a graph.

1. A waterwheel, that supplies a mill, has a diameter of 40 feet. You start your stopwatch at a point P, which is on the circumference of the waterwheel and on an axis through the center of the waterwheel, is parallel to the surface of the water. The wheel turns counter-clockwise at a constant rate. Two seconds later, point P on the rim of the wheel is at its greatest height. The center of the waterwheel is located 15 feet above the water and 15 feet to the right of the mill which is our designated frame of reference. You are to model the distance d of point P from the surface of the water in terms of the number of seconds t.

2. As you ride a Ferris wheel moving counter-clockwise at a constant rate, your distance from the ground varies sinusoidally with time. You are the last seat to be filled at the bottom of the Ferris wheel and then it starts. Let your seat be point P located on the circumference of the Ferris wheel. Let t be the number of seconds that have elapsed since the Ferris wheel started. You find that it takes you 15 seconds for point P to reach the top, 23 feet above the ground. The diameter of the wheel is 20 feet. The center of the Ferris wheel is located 30 feet to the right of our designated frame of reference. You are to model the distance d of point P from the ground.

Inverse Trigonometric Functions

In Unit 1, we defined an inverse function as one that switches the domain and range of the parent function. Recall that not all functions are invertible. That is, not all functions have inverses that are functions. A function must be one-to-one (invertible) to have an inverse function. None of the common trigonometric functions: sine, cosine, and tangent are injective. Thus, their domains must be restricted to make them one-to-one such that their inverses are defined. Once the common trigonometric functions are restricted to their *principal branches* the consequent function inverses become very useful. Similar to the exponent and logarithm relationship described in Unit 3, one can express the function inverse relationship between the trigonometric functions and their inverses in a general way as follows using the sine function as an example

$$\sin(\text{angle}) = \text{answer}$$

implies that

$$\sin^{-1}(\text{answer}) = \text{angle}$$

Notice that the inverse or "arc" trigonometric functions are symbolized as $\sin^{-1}(x)$, $\cos^{-1}(x)$, and $\tan^{-1}(x)$. There are also inverse functions for the cosecant, secant, and cotangent functions after restriction, but we will concentrate our efforts on the more common sine, cosine, and tangent functions along with their inverses.

Exploration 4.3.6: Common Inverse Trigonometric Functions

1. Sketch the graph of $f(x) = \sin x$ from $-2\pi \leq x < 2\pi$.
2. Explain why $f(x)$ does not have an inverse function.
3. How might you modify or restrict $f(x)$ such that it does have an inverse function?
4. Consider *cosine* and *tangent*. Do they have an inverse function? How might you modify or restrict them with an interval around the origin such that they have an inverse?

Use your conjectures from above to complete the table below.

Function	**Domain**	**Range**
$y = \sin^{-1} x$		
$y = \cos^{-1} x$		
$y = \tan^{-1} x$		

5. Based on what you've learned, answer $a - e$.

a. $\cos^{-1}(1) =$

b. $\tan^{-1}(-\sqrt{3}) =$

c. $\cos^{-1}\left(\cos\dfrac{\pi}{3}\right) =$

d. $\cos^{-1}\left(\cos\dfrac{4\pi}{3}\right) =$

e. $\sin\left(\tan^{-1}\dfrac{4}{3}\right) =$

Extension Challenge: The domains of the $\csc(x)$, $\sec(x)$, and $\cot(x)$ trigonometric functions can also be restricted such that they are invertible. Do some research to find information about the inverses of these functions and their properties.

Lesson 4.4: Angular and Linear Speed

We will now explore an application of trigonometric properties in relation to *angular and linear speed*. All of the information that you need to know about this topic is contained within the following Exploration.

Exploration 4.4.1: Angular and Linear Speed - *Can You Feel the Breeze?*

The coordinates of Austin, Texas, are $30.27°$N and $97.76°$W. Using 3960 miles as the radius of Earth, we want to find the linear speed of a person living in Austin, with respect to the center of Earth, in miles per hour.

In answering the question, we will consider *linear speed*, v_1, to be the distance traveled along an arc (the arclength), s, per unit of time, t. We will also consider *angular speed*, w , to be the radians traveled that correspond to the arclength, θ, per unit of time, t.

Background:

1. Write an equation for linear speed, v_1, in terms of angular speed w, using the fact that the arclength is equal to the radius of the circle, r, times the angle, θ, corresponding to the arc $(s = r\theta)$.

Before we attempt to answer the title question, let's first consider a slightly simpler question. We will consider a merry-go-round, a circle, spinning around an axis rather than the earth, which is a sphere.

Merry-Go-Round

Suppose a merry-go-round has a diameter of 10 feet and revolves once per minute.

2. Find the angular speed of the merry-go-round in radians per minute.

3. If Olivia is standing 3 feet from center of the merry-go-round, what distance does she travel in one revolution?

4. If Libby is standing 4 feet from the center, what distance does she travel in one revolution?

5. Who is traveling faster? Olivia or Libby? Justify your answer by finding the linear speeds of Olivia and Libby, with respect to the center of the merry-go-round.

6. Is there a place where the linear speed of a person on the merry-go-round is 0 feet per minute? Where should a person stand so that they have the fastest linear speed?

Earth

Now let's consider the question of a person standing at a specified location on Earth.

7. What is the angular speed of Earth, in radians per hour?

8. Suppose Libby lives in Pontianak, Indonesia, right on the equator. What is Libby's linear speed, with respect to the center of Earth, in miles per hour?

9. Is there a place on Earth where a person has a linear speed of 0 miles per hour?
   Explain your answer.

10. Suppose Olivia lives in Austin. Do you think Olivia has the same linear speed as Libby or is she going faster or slower?
    Explain your answer.

11. Find the radius of the latitudinal circle on which Olivia, in Austin, travels. [**Note:** Since latitude is measured in degrees, you should solve this problem using degrees.]

12. Find Olivia's linear speed, with respect to the center of Earth, in miles per hour. Did you guess correctly in problem 10?

13. You should have found that Olivia and Libby are going very fast — twice as fast as an airplane. Why don't we feel a breeze as a result of the earth's rotation?

Historical Note:

The mathematician Archimedes took the question of this Exploration one step further by studying spirals. We've seen in this Exploration that if you stand on a merry-go-round while it's turning, then you travel in a circle. Imagine what happens if, instead of standing still, you walk from the center out to the edge of the merry-go-round at a constant speed while it's turning. The path you travel is no longer a circle, but along a path known as the *Spiral of Archimedes.*

Archimedes, born in about 287 BCE, was the most famous mathematician and inventor in ancient Greece. Among his contributions is an invention known as the Archimedes Screw that is still used today to pump water from the ground. In 218 BCE, when Archimedes was 69 years old and living in the city of Syracuse, the Second Punic War began. By 214 BCE the Romans began a siege of the city of Syracuse. As the Romans were capturing the city in 212 BCE, after the two-year-long siege, a soldier killed Archimedes. He was 75 years old. His last words are said to have been, "Don't disturb my circles."

Lesson 4.5: Back to Identities

At this stage of your study of trigonometric functions you have learned a geometric definition of each of these functions and properties of their graphs. In applications of trigonometric functions, expressions such as $\sin(\varphi + \theta)$ and $\cos(\varphi + \theta)$ are frequently encountered. Our study of trigonometry continues with the discovery of expressions for the sine and cosine of sums and differences of angles and for half and double angles. These types of expressions are part of analytic trigonometry and they are used extensively throughout the sciences, further mathematical studies, and to simplify complex statements in economics.

The chart on page 106 summarizes many of the most important trigonometric identities. These should be memorized, but it will help you to understand these identities if you also know how they arose and how they are derived. Exploration 4.5.1 will shed some light on this subject. Note that it may be helpful to refer to this chart as you do Exploration 4.5.1.

Exploration 4.5.1: Investigating Trigonometric Identities

1. State the symmetry of the graphs of sine, cosine, and tangent.

2. Determine whether each trigonometric function is even or odd. Justify your answers. [**Hint:** recall that $f(-x) = f(x)$ for even functions and $f(-x) = -f(x)$ for odd functions.]

3. Use the results from problem 2 to determine the Even/Odd Identities for sine, cosine, and tangent.

4. Sketch the graph of the sine function after a translation to the right of $\frac{\pi}{2}$. How is this graph related to the graph of the cosine function?

5. Sketch the graph of the cosine function after a translation to the right of $\frac{\pi}{2}$. How is this graph related to the graph of the sine function?

6. Using your sketches, derive the Cofunction Identities:

$$\sin\left(x - \frac{\pi}{2}\right) =$$

$$\cos\left(x - \frac{\pi}{2}\right) =$$

Exploration 4.5.2: Yes, More Trigonometric Identities

Background:

1. Draw the reference triangle in the Unit Circle with θ as $\frac{\pi}{12}$.

2. Estimate an approximate value of $\cos\left(\frac{\pi}{12}\right)$. [**Hint:** Remember the ratio definition of cosine.]

3. One might reason since $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$, then $\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{6}\right)$. Is this statement true? Show work to support your answer.

Exploration:

We begin this Exploration by finding a formula for $\cos(\varphi - \theta)$ with restrictions $0 \leq \theta \leq \varphi < \pi$. Consider the figures below when completing the exercises.

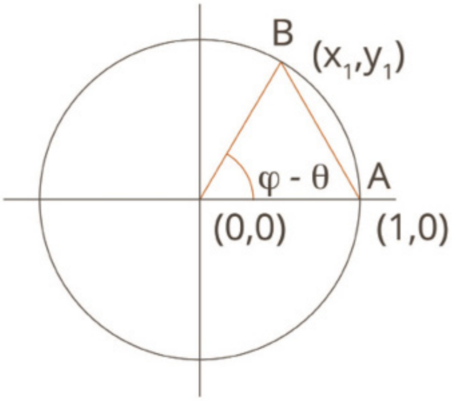


Figure 4.5.2-1

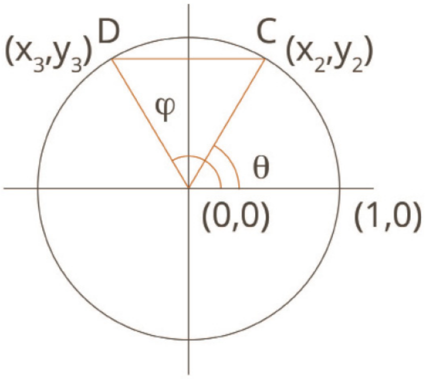


Figure 4.5.2-2

Figure 4.5.2-1 is the graph of an angle $(\varphi - \theta)$ in standard position with the Unit Circle centered at $(0,0)$. $\overline{AB}$ is the chord opposite the angle $(\varphi - \theta)$.

Figure 4.5.2-2 is the graph of the separate angles, φ and θ, in standard position with the Unit Circle centered at $(0,0)$. $\overline{CD}$ is the chord opposite the angle $(\varphi - \theta)$.

4. Given the two inscribed triangles are congruent, write expressions for the lengths of the chords in the two diagrams provided using the coordinates of the end-points of each chord.

5. Use the results of Exercise 4 to derive an expression for $\cos(\varphi - \theta)$. [**Hint:** Replace the Cartesian coordinates with their equivalents in terms of sines and cosines.]

6. Use the result from Exercise 5 to find an algebraic expression for $\cos\left(\frac{\pi}{12}\right)$. Evaluate this expression and compare to a calculated or tabulated value.

7. Use the result from Exercise 5 to find a formula for $\cos(\varphi + \theta)$ in terms of trigonometric functions of φ and θ.

8. Make a conjecture as to how one could rewrite (2φ) in order to derive identities using the sum and difference formulas in terms of $\cos\varphi$. Then, find a formula for $\cos(2\varphi)$.

Challenge:

9. Find the formulas for $\sin(\varphi + \theta)$ and $\sin(\varphi - \theta)$. [**Hint:** Consider the *cofunction trigonometric identities*.]

10. Find a formula for $\sin(2\varphi)$, then determine two more formulas for $\cos(2\varphi)$

11. Use the results above to find an algebraic expression for $\cos\left(\frac{8\pi}{3}\right)$. Evaluate this expression by using one of the double-angle formulas derived previously. Verify the value you just calculated with the value of the cosine of the co-terminal angle on the Unit Circle.

The Law of Cosines and the Law of Sines

For any ΔABC with sides a, b, and c and with opposite angles A, B, and C respectively:

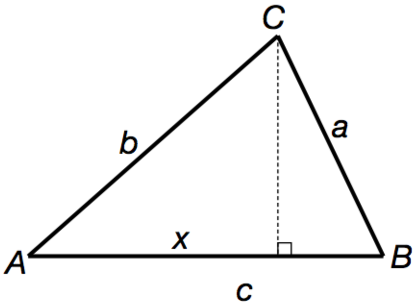
$$c^2 = a^2 + b^2 - 2ab\cos C \text{ (Law of Cosines)}$$

where θ is the angle between a and b. Moreover,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \text{ (Law of Sines)}$$

Exploration 4.5.3: The Laws

1. Derive the Law of Cosines and the Law of Sines (we will consider only the acute triangle case).



2. Note that the Law of Cosines is a *stronger statement* of the Pythagorean Theorem. What do you think is meant by '*stronger statement*' in this case?

3. It is important to note that the Law of Sines may provide you with one answer, more than one answer, or no answer if a non-included angle and two sides of a triangle are used to find a second angle within a given triangle. This is called the Ambiguous Case of the Law of Sines. The Ambiguous Case is directly related to the fact that there is no Side, Side, Angle (SSA) triangle congruence criterion. Can you explain this relationship?

Use the law of Sines to solve:

4. A 92 ft path cuts diagonally from Mateo St. to Cove St. It makes an angle of $68°$ with Cove St. and an angle of $34°$ with Mateo St. Determine the distance saved by using the path versus walking along the streets.

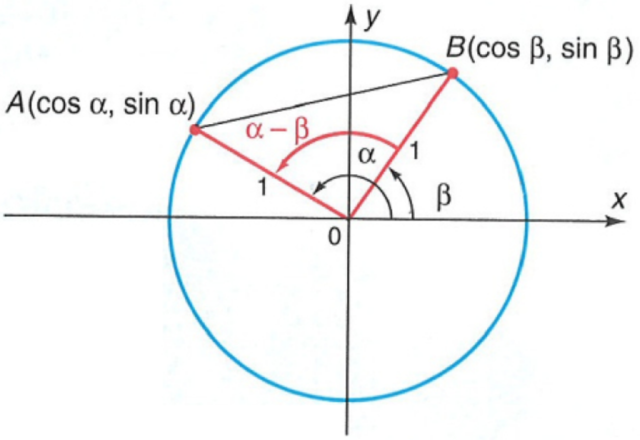
5. The angle of depression from a police helicopter to its landing pad is $56°$. If the helicopter is flying at 1000 ft, determine the distance from the police helicopter to the landing pad.

Use the Law of Cosines to solve.

6. Two flying drones leave an airport at the same time. The heading of the first is $120°$ and the heading of the second $320°$. If the drones are traveling at 80 and 50 mph respectively, how far apart will they be after 1 hour?

7. If two forces act on a point, the resultant force (or magnitude of their sum) is the length of the diagonal of a parallelogram that includes the common point at the vertex. The lengths of the adjacent sides of the parallelogram represent the magnitudes of the two respective forces. If forces of 250 lbs and 170 lbs act at an angle of $46°$ to each other, determine the magnitude of their sum.

8. Use the Law of Cosines and the Distance Formula to derive the trigonometric expansion identity for $\cos(\alpha - \beta)$. This picture may help.



Lesson 4.6: The Roller Coaster

Exploration 4.6.1 will give you a chance to pull together much of your knowledge about trigonometric functions to this point.

Exploration 4.6.1: Building a Roller Coaster

An established company, the Thrill Ride Roller Coaster Company, has asked your group to help design a roller coaster track.

The Thrill Ride Company wants to build a roller coaster subject to a set of constraints. You are told that the company has several engineers who could design a blueprint and build the track if only they knew the functions whose graphs would define the desired curve of the track. According to a company spokesperson, her people can easily “fit” a curve to a set of points; however, the resulting curve does not necessarily satisfy constraints involving slopes, concavity, extreme values, smoothness, updateability, etc. Furthermore the company’s engineers must first build a scaled-down test model, and thus they need the function description for the scale model. Hence your task is to define a *piecewise* function, that is a function defined in pieces over the interval $[0, 15]$, whose graph satisfies the following constraints for the roller coaster track (each unit represents 10 meters).

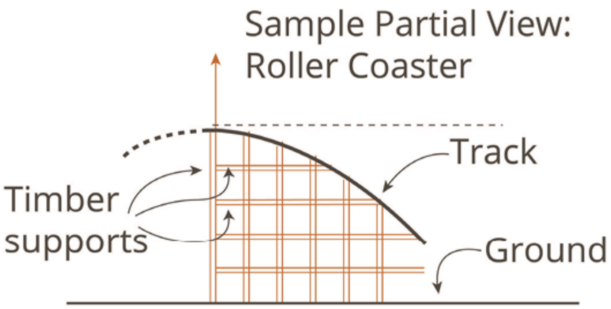
1. The entrance onto the track is at the point $(0, 10)$ and the exit is at $(15, 0)$. There are just two local extreme values, a minimum at $(4, 2)$ and a maximum at $(8, 8)$.
   (You do not have to consider designing the stairs leading to the entrance.)

2. The slope of the curve at the entrance and exit points must be zero in order to facilitate getting on and off the roller coaster car.

3. **IMPORTANT:** The curve must be *smooth*, meaning that the piecewise function must knit together without any "kinks" in the track over its entire domain.

4. For continued customer interest, maintenance, and future customization, **YOU MUST** build the roller coaster track out of pieces. Based on the given constraints, you must also decide how many functions you will "knit together" to build the track.

The Thrill Ride Company is skeptical of work that has not been refined, stating that whenever safety, cost, or time is involved they will not accept the plans of a single person. Therefore, you are to work in a group of three or four people.



A SUMMARY OF SOME IMPORTANT TRIGONOMETRIC IDENTITIES

Fundamental Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{cos\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Cofunction Identities

$\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$ and $\sin\left(\theta - \frac{\pi}{2}\right) = -\cos\theta$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

Odd and Even Identities

$$\begin{array}{ll} \sin(-\theta) = -\sin\theta & \csc(-\theta) = -\csc(\theta) \\ \cos(-\theta) = \cos\theta & \sec(-\theta) = \sec(\theta) \\ \tan(-\theta) = -\tan\theta & \cot(-\theta) = -\cot(\theta) \end{array}$$

Addition and Subtraction Identities

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

Double Angle Identities

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \end{aligned}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$